**Goal:** Discretize $2^{nd}$ order linear differential eqn

i.e. convert differential eqn

$$y'' + p(t)\,y' + q(t)\,y = f(t)$$

to matrix eqn

$$D\,\underline{y} = \underline{f}$$

$D$ = matrix of numbers

$\underline{f}$ = vector of numbers

$\underline{y}$ = vector of unknowns.

First we will discretize using pointwise formulas

→ $y_k'$ = centered difference

$$= \tfrac{1}{2h}\left(y_{k+1} - y_{k-1}\right)$$

→ $y_k'' = ??$

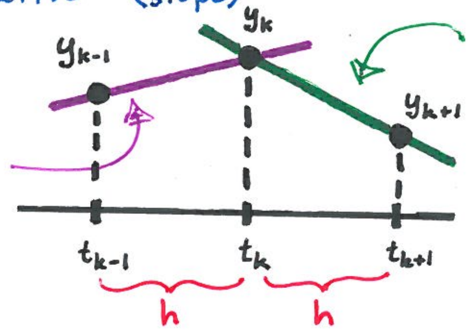
**Pointwise $2^{nd}$ Derivative Formula.**

$$\boxed{y_k'' = \tfrac{1}{h^2}\left(y_{k+1} - 2y_k + y_{k-1}\right)}$$

## $\frac{d^2}{dt^2}$ Formula.

Second derivative is change in first derivative (slope)

Slope before $t_k$ is $\dfrac{y_k - y_{k-1}}{h}$




Slope after $t_k$ is $\dfrac{y_{k+1} - y_k}{h}$

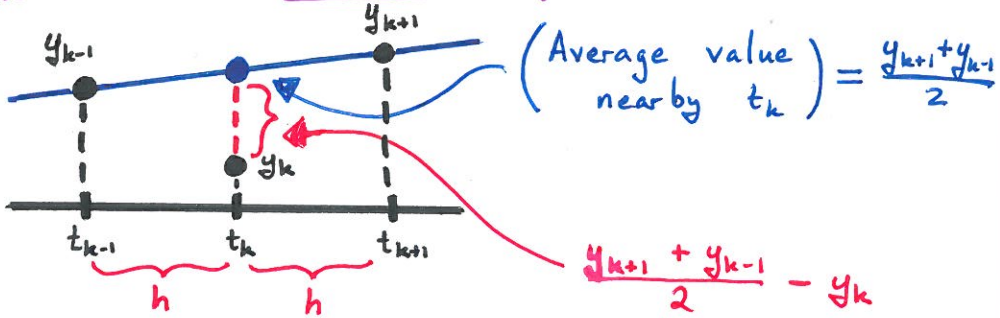
$$\frac{\text{Change in slope at } t_k}{h} = \frac{\left(\frac{y_{k+1} - y_k}{h}\right) - \left(\frac{y_k - y_{k-1}}{h}\right)}{h}$$

$$= \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2}$$

**Note:** This formula is a backward diff. of forward differences

$$y_k'' = \frac{\frac{y_{k+1} - y_k}{h} - \frac{y_k - y_{k-1}}{h}}{h} = \frac{y_k' - y_{k-1}'}{h}$$

**Remark:** An alternate way to understand $\frac{d^2}{dt^2}$ is to note that it measures difference between $y_k$ and average value nearby




$$y_k'' = \frac{1}{h^2/2}\left(\frac{y_{k+1}+y_{k-1}}{2} - y_k\right)$$

$$= \frac{2}{h^2}\left(\frac{y_{k+1}+y_{k-1}-2y_k}{2}\right)$$

$$= \frac{1}{h^2}\left(y_{k+1} - 2y_k + y_{k-1}\right)$$

**EX:** Compute $\frac{d^2}{dt^2}$ of $\underline{\underline{f}} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}$ with $h = 1/2$.

$\underline{\underline{f}}$

$f_0 = 1 \leadsto f_0'' = \frac{1}{h^2}(f_1 - 2f_0 + f_{-1})$ cannot be computed

$f_1 = 0 \leadsto f_1'' = \frac{1}{h^2}(f_2 - 2f_1 + f_0) = 4(-1-0+1) = 0$

$f_2 = -1 \leadsto f_2'' = \frac{1}{h^2}(f_3 - 2f_2 + f_1) = 4(-2+2+0) = 0$

$f_3 = -2 \leadsto f_3'' = \frac{1}{h^2}(f_4 - 2f_3 + f_2) = 4(1+4-1) = 16$

$f_4 = 1 \leadsto f_4'' = \frac{1}{h^2}(f_5 - 2f_4 + f_3)$ cannot be computed

$$\underline{\underline{f''}} = \begin{bmatrix} 0 \\ 0 \\ 16 \end{bmatrix}$$

In matrix form

$$\begin{array}{c} \cancel{f_0''} \\ \begin{bmatrix} f_1'' \\ f_2'' \\ f_3'' \\ \vdots \\ f_n'' \end{bmatrix} \\ \cancel{f_{n+1}''} \end{array} = \frac{1}{h^2}\underbrace{\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -2 & 1 & \cdots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & 1 & -2 & 1 \end{bmatrix}}_{2^{\text{nd}}\text{ Derivative Matrix.}} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \\ f_{n+1} \end{bmatrix}$$

**EX** Compute $f''$ for $f = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ with $h = 1/3$.

$$\begin{bmatrix} f_1'' \\ f_2'' \\ f_3'' \end{bmatrix} = \frac{1}{(1/3)^2}\begin{bmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

$$= 9\begin{bmatrix} 1-4+0 \\ 2-0-1 \\ 0+2+2 \end{bmatrix} = 9\begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$$

We can use the pointwise second derivative formula to discretize $2^{nd}$ order differential eqns.

$$y_k'' = \frac{1}{h^2}\left(y_{k+1} - 2y_k + y_{k-1}\right)$$

**EX:** Discretize

$$-y'' = 4t - 1 \quad \text{with } y(1) = 0,\ y(3) = 0$$

on $[1,3]$ with $h = 1/2$.

**Note:** Since this is a $2^{nd}$ order DE (with $y''$) solving requires two integrations, yielding two unknown constants. Problem must include two values.

- at same point $\begin{Bmatrix} y(a) = \cdots \\ y'(a) = \cdots \end{Bmatrix}$ "initial value" problem (IVP)
- at different points $\begin{Bmatrix} y(a) = \cdots \\ y(b) = \cdots \end{Bmatrix}$ "boundary value" problem (BVP)

Boundary value problems are more difficult to solve, so we will focus on discretizing them.

| $t$ | $y$ | |
|---|---|---|
| $t_0 = 1$ | $y_0 = y(t_0) = y(1) = 0$ | Boundary value determines first $y$ $(y_0 = 0)$ |
| $t_1 = 3/2$ | $y_1$ | |
| $t_2 = 2$ | $y_2$ | |
| $t_3 = 5/2$ | $y_3$ | |
| $t_4 = 3$ | $y_4 = y(t_4) = y(3) = 0$ | Boundary value determines last $y$ $(y_{n+1} = 0)$ |

$$-y'' = 4t - 1$$

unknown $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ plugs into DE $\Rightarrow$ $\begin{Bmatrix} -y_1'' = 4t_1 - 1 \\ -y_2'' = 4t_2 - 1 \\ -y_3'' = 4t_3 - 1 \end{Bmatrix}$

$$\begin{Bmatrix} -\frac{1}{h^2}(y_2 - 2y_1 + \cancel{y_0}) = -y_1'' = 4t_1 - 1 = 4\cdot\frac{3}{2} - 1 = 5 \\ -\frac{1}{h^2}(y_3 - 2y_2 + y_1) = -y_2'' = 4t_2 - 1 = 4\cdot 2 - 1 = 7 \\ -\frac{1}{h^2}(\cancel{y_4} - 2y_3 + y_2) = -y_3'' = 4t_3 - 1 = 4\cdot\frac{5}{2} - 1 = 9 \end{Bmatrix}$$

$y_0 = 0$

$y_4 = 0$

$$\frac{1}{h^2} = \frac{1}{(1/2)^2} = \frac{1}{1/4} = 4$$

$$\begin{Bmatrix} -4y_2 + 8y_1 = 5 \\ -4y_3 + 8y_2 - 4y_3 = 7 \\ 8y_3 - 4y_4 = 9 \end{Bmatrix} \Longrightarrow \boxed{\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}}$$

$$\left(\underline{\text{alt.}} \quad \frac{1}{(1/2)^2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}\right)$$

Since we know $y_0$ & $y_{n+1}$ from boundary values, we can use <u>centered differences</u> for $y'$ in boundary value problems.

<u>EX</u>: Discretize

$$y'' + 2y' = t + 1 \quad \text{with} \quad y(\underline{2}) = 0 \quad y(\underline{3}) = 0$$

using step-size $h = 1/5$

(<u>Note</u>: The interval should always be between the two boundary points – in this case $[2, 3]$.)

$$\begin{array}{ll} \underline{t} & \underline{y} \\ t_0 = 2 & y_0 = 0 \\ t_1 = 11/5 & \\ t_2 = 12/5 & \text{unknown} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \\ t_3 = 13/5 & \\ t_4 = 14/5 & \\ t_5 = 3 & y_5 = 0 \end{array}$$

$$\underline{y'' \quad + \quad 2\,y' \quad = \quad t+1}$$

$$\begin{cases} \frac{1}{h^2}(y_2 - 2y_1 + \cancelto{0}{y_0}) + 2\,\frac{1}{2h}(y_2 - \cancelto{0}{y_0}) = 11/5 + 1 \\ \frac{1}{h^2}(y_3 - 2y_2 + y_1) + 2\,\frac{1}{2h}(y_3 - y_1) = 12/5 + 1 \\ \frac{1}{h^2}(y_4 - 2y_3 + y_2) + 2\,\frac{1}{2h}(y_4 - y_2) = 13/5 + 1 \\ \frac{1}{h^2}(\cancelto{0}{y_5} - 2y_4 + y_3) + 2\,\frac{1}{2h}(\cancelto{0}{y_5} - y_3) = 14/5 + 1 \end{cases}$$

(Combine terms)

$$1/h^2 = 1/(1/5)^2 = 1/(1/25) = 25 \qquad 1/h = 1/(1/5) = 5$$

$$\begin{cases} (25+5)\,y_2 + (-50)\,y_1 = 16/5 \\ (25+5)\,y_3 + (-50)\,y_2 + (25-5)\,y_1 = 17/5 \\ (25+5)\,y_4 + (-50)\,y_3 + (25-5)\,y_2 = 18/5 \\ (-50)\,y_4 + (25-5)\,y_3 = 19/5 \end{cases}$$

$$\boxed{\begin{bmatrix} -50 & 30 & 0 & 0 \\ 20 & -50 & 30 & 0 \\ 0 & 20 & -50 & 30 \\ 0 & 0 & 20 & -50 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 16/5 \\ 17/5 \\ 18/5 \\ 19/5 \end{bmatrix}}$$

<u>alt</u>:

$$\Bigg( \underbrace{\frac{1}{(1/5)^2} \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}}_{d^2/dt^2} \underbrace{+\, 2}_{+\,2} \underbrace{\frac{1}{2(1/5)} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{d/dt} \Bigg) \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 16 \\ 17 \\ 18 \\ 19 \end{bmatrix}$$

For $2^{nd}$ order differential equations, the (<u>fixed</u>-<u>fixed</u>) <u>second derivative</u> <u>matrix</u> is

$$\frac{1}{h^2} \begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & & \vdots \\ 0 & 1 & -2 & 1 & & \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & 1 \\ 0 & \cdots & & 0 & 1 & -2 \end{bmatrix}$$

<u>EX</u>:

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \qquad \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \qquad \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

For $2^{nd}$ order differential equations, the (fixed-fixed) first derivative matrix is

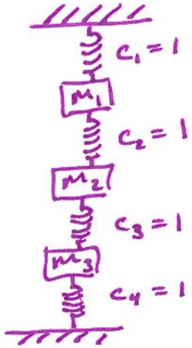
$$\frac{1}{2h}\begin{bmatrix}0&1&0&0&\cdots&0\\-1&0&1&0&&\vdots\\0&-1&0&1&&0\\\vdots&&\ddots&\ddots&\ddots&1\\0&\cdots&&0&-1&0\end{bmatrix}$$

$$\left(\begin{bmatrix}0&1\\-1&0\end{bmatrix}\quad \text{EX:}\ \begin{bmatrix}0&1&0\\-1&0&1\\0&-1&0\end{bmatrix}\quad \begin{bmatrix}0&1&0&0\\-1&0&1&0\\0&-1&0&1\\0&0&-1&0\end{bmatrix}\right)$$

**Note:** The (fixed-fixed) negative $2^{nd}$ derivative $\left(-\frac{d^2}{dt^2}\right)$ matrix is the stiffness matrix for a spring system!

$$K = -\begin{bmatrix}-2&1&0\\1&-2&1\\0&1&-2\end{bmatrix}$$

$$= \begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}$$ stiffness matrix of




(line of masses with fixed ends)

**EX:** Discretize $y'' + ty = \delta(t)$ $\quad y(-1)=0 \quad y(1)=0$ with step-size $h = 1/2$.

| $t$ | $y$ |
|---|---|
| $t_0=-1$ | $y_0=0$ |
| $t_1=-\frac{1}{2}$ | $y_1$ |
| $t_2=0$ | $y_2$ |
| $t_3=\frac{1}{2}$ | $y_3$ |
| $t_4=1$ | $y_4=0$ |

unknown $\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$

$$y'' + t\,y = \delta(t-0)$$

$$\begin{cases}\frac{1}{h^2}(y_2-2y_1+\cancel{y_0}^{\,0}) + (-\frac{1}{2})y_1 = 0\\ \frac{1}{h^2}(y_3-2y_2+y_1) + (0)\,y_2 = \frac{1}{h}\\ \frac{1}{h^2}(\cancel{y_4}^{\,0}-2y_3+y_2) + (\frac{1}{2})\,y_3 = 0\end{cases}$$

Impulse Point

**Note:** $\frac{1}{h^2} = \frac{1}{(1/2)^2} = \frac{1}{1/4} = 4$

$\frac{1}{h} = \frac{1}{(1/2)} = 2$

$$\begin{cases}4y_2 + (-8-\frac{1}{2})y_1 = 0\\ 4y_3 + (-8+0)y_2 + 4y_1 = 2\\ (-8+\frac{1}{2})y_3 + 4y_2 = 0\end{cases}$$

$$\boxed{\begin{bmatrix}-8-\frac{1}{2}&4&0\\4&-8&4\\0&4&-8+\frac{1}{2}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = \begin{bmatrix}0\\2\\0\end{bmatrix}}$$

**alt:**

$$\left(\underbrace{\frac{1}{(1/2)^2}\begin{bmatrix}-2&1&0\\1&-2&1\\0&1&-2\end{bmatrix}}_{d^2/dt^2} + \underbrace{\begin{bmatrix}-\frac{1}{2}&0&0\\0&0&0\\0&0&\frac{1}{2}\end{bmatrix}}_{t}\right)\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = \begin{bmatrix}0\\2\\0\end{bmatrix}$$

For $2^{nd}$ order differential equations, the "multiply by $p(t)$" matrix is

$$\begin{bmatrix} p(t_1) & 0 & 0 & \cdots & 0 \\ 0 & p(t_2) & 0 & \cdots & 0 \\ 0 & 0 & p(t_3) & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & p(t_n) \end{bmatrix}$$

## Types of Boundary Values.

- A boundary value problem with conditions

$$\begin{cases} y(a) = \text{something} \\ y(b) = \text{something} \end{cases}$$

is called "fixed-fixed" because the boundary conditions hold the endpoints of $y$ still ("fixing" them in place) no matter what the forcing function $f(t)$ is.

- A boundary value problem with conditions

$$\begin{cases} y(a) = \text{something} \\ y'(b) = \text{something} \end{cases}$$

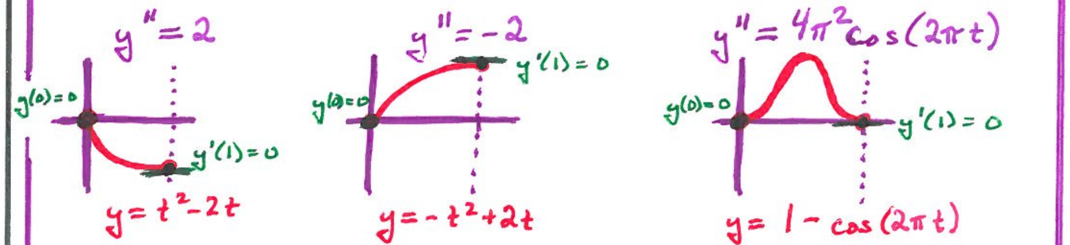
is called "fixed-free" because different forcing functions could change the right endpoint $y(b)$

| | |
|---|---|
| $\left.\begin{matrix} y(a)=0 \\ y(b)=0 \end{matrix}\right\}$ "fixed fixed" | $\left.\begin{matrix} y(a)=0 \\ y'(b)=0 \end{matrix}\right\}$ "fixed free" |
| $\left.\begin{matrix} y'(a)=0 \\ y(b)=0 \end{matrix}\right\}$ "free fixed" | $\left.\begin{matrix} y'(a)=0 \\ y'(b)=0 \end{matrix}\right\}$ "free free" |

The same notation is used when discussing variations of the heat and wave equation in MAT 219 ("free" $\longleftrightarrow$ "insulated end")

Solutions to "fixed-free" BVP

$$y'' = f(t) \qquad y(0) = 0 \qquad y'(1) = 0$$




- $y(0)$ is "fixed" at 0
- but $y(1)$ is "free" to move